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CHAINS IN CR GEOMETRY

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Abstract

There is a well-defined system of curves on any nondegenerate CR manifold of hypersurface type. It is shown that any two sufficiently close points on a strictly pseudo-convex abstract CR hypersurface can be connected by a smooth curve from this family. Such a general result does not hold for other signatures.

This paper shows that any two sufficiently close points on a strictly pseudo-convex CR hypersurface can be connected by a smooth chain. For purposes of exposition we first present the complete proof for a three-dimensional submanifold of $\mathbf{C}^2$. Then in §4 we indicate the changes necessary to cover the general case of an abstract CR structure of hypersurface type. We work with C^∞ structures but it will be obvious that one only needs C^k, k large. The results are new even for real analytic structures and it is not clear if a shorter proof would be possible in this case.

A CR structure on a three-dimensional manifold M is a 2-plane distribution $H \subset TM$ together with a fibre preserving map J: $H \to H$ with $J^2 = -I$. Given such a structure, we may choose a real 1-form ω which annihilates H and a complex 1-form ω_1 which annihilates all vectors of the form $X + iJX$, $X \in H$. These choices can be done in such a way that $\omega \wedge \omega_1 \wedge \bar{\omega}_1$ is different from zero in a neighborhood of a given point. We are interested in results of a local nature on M so we may shrink M and assume $\omega \wedge \omega_1 \wedge \bar{\omega}_1$ is everywhere different from zero. Conversely, given ω and ω_1 with $\omega \wedge \omega \wedge \bar{\omega}_1 \neq 0$ we may easily construct H and J.

Any three-dimensional submanifold M of $\mathbf{C}^2$ has an induced CR structure. Let $\tilde{J}$: $T\mathbf{C}^2 \to T\mathbf{C}^2$ give the complex structure. Then $H = TM \cap \tilde{J}TM$ and $J = \tilde{J}|_H$. Note that if Φ: $U \to V$ is a biholomorphism of open sets in $\mathbf{C}^2$, then $M \cap U$ and $\Phi(M) \cap V$ have the same CR structure. The forms ω and ω_1 can

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also be directly determined for $M \subset \mathbf{C}^2$. To do this let M be given by $r(Z, \bar{Z}) = 0$ with $Z = (z_1, z_2)$ and $dr \neq 0$ at points of M. For definiteness assume that at a given point $p \in M$ we have $dz_1 \wedge \partial r \neq 0$. Now let $\omega = i\partial r$ and $\omega_1 = dz_1$. Thus $\omega \wedge \omega_1 \wedge \bar{\omega}_1 \neq 0$. As a form on M, $dr = 0$ and so $\partial r = -\bar{\partial} r$. Thus $i\partial r$ is a real form. It is easy to show that this gives the same CR structure as the previously defined H and J.

We will be interested only in CR structures generic in the following sense.

Definition. The CR structure on M^3 is strictly pseudo-convex if $d\omega = ig\omega_1 \wedge \bar{\omega}_1 \pmod{\omega}$ with $g \neq 0$.

It is easily seen that if this condition holds for one choice of ω and ω_1 for the given CR structure it holds for all choices. Given such an M^3 in $\mathbf{C}^2$ we can find a local biholomorphism which puts M into a nice form.

For a function $g(x)$, $x \in \mathbf{R}^n$, defined near the origin let us use $g \in \mathcal{O}(N)$ to mean $g \in C^\infty$ and $D^k g(0) = 0$ for $0 \leqslant |k| \leqslant N - 1$. Here k is a multi-index and D^k stands for all partial derivatives of order less than or equal to $N - 1$. (Thus we do *not* use $\mathcal{O}(1)$ to denote a bounded function.) Note that if $g \in \mathcal{O}(N)$, then $g' \in \mathcal{O}(N - 1)$ for any first derivative of g.

Lemma. *Let $M^3 \subset \mathbf{C}^2$ with M strictly pseudo-convex at the point p. There exists a biholomorphism Φ of a neighborhood of p onto a neighborhood of the origin such that $\Phi(p) = 0$ and $\Phi(M)$ is given by*

$$\{(z_1, z_2) \colon \operatorname{Im} z_2 = |z_1|^2 + G(z_1, \bar{z}_1, \operatorname{Re} z_2),\ G \in \mathcal{O}(4)\}. \tag{0.1}$$

Proof. Find some real analytic hypersurface M' which is tangent to M at p to order N. This means that as graphs over the common tangent plane, M and M' are given by functions f and f' with $f - f' \in \mathcal{O}(N + 1)$ where p is considered as the origin. As shown in [3] there is a local biholomorphism Φ such that

$$\Phi(M') = \left\{(z_1, z_2) \colon \operatorname{Im} z_2 = |z_1|^2 + \sum_{\substack{j \geqslant 2 \\ k \geqslant 2}} c(u) z^j \bar{z}^k \right\}.$$

Thus

$$\Phi(M) = \left\{(z_1, z_2) \colon \operatorname{Im} z_2 = |z_1|^2 + \sum_{\substack{j \geqslant 2 \\ k \geqslant 2}} c(u) z^j \bar{z}^k + g,\ g \in \mathcal{O}(N + 1)\right\}.$$

This proves the Lemma, as long as N was chosen greater than two.

Let us set $z_1 = z$ and $z_2 = u + iv$. Then (z, u) provide local coordinates on any hypersurface of the form (0.1). The simplest such hypersurface is given by $v = |z|^2$. This submanifold is called the hyperquadric and is denoted by Q. As a defining function we take $r = (z_2 - \bar{z}_2)/2i - z_1\bar{z}_1$. As before let $\omega_1 = dz$ and $\omega = i\partial r$. Expanding the latter we see

$$\omega = \tfrac{1}{2}\, dz_2 - \bar{z}_1\, dz_1 = \tfrac{1}{2} d\left(u + i|z|^2\right) - \bar{z}_1\, dz_1$$

and so

$$\omega = \tfrac{1}{2}(du + iz\, d\bar{z} - i\bar{z}\, dz). \tag{0.2}$$

Note that

$$d\omega = i\omega_1 \wedge \bar{\omega}_1. \tag{0.3}$$

In particular, Q is strictly pseudo-convex.

Now let us find ω for the induced CR structure on a submanifold of the form given previously:

$$v = |z|^2 + G, \qquad G = \mathcal{O}(4). \tag{0.4}$$

Take for the defining function

$$r = \frac{1}{2i}(z_2 - \bar{z}_2) - z_1\bar{z}_1 - G,$$

where G is extended off of the (z, u)-plane to be independent of v. Then

$$\omega = i\partial r = \tfrac{1}{2}\left((1 + A)\, du + (iz + B)\, d\bar{z} - (i\bar{z} - \bar{B})\, dz\right)$$

with $A = G_u^2 = \mathcal{O}(6)$ and $B = \mathcal{O}(3)$.

We replace ω by a real multiple to achieve

$$\omega = \tfrac{1}{2}\left(du + (iz + C)\, d\bar{z} - (i\bar{z} - \bar{C})\, dz\right)$$

with $C = \mathcal{O}(3)$.

Note that

$$d\omega = i\, dz \wedge d\bar{z} + \tfrac{1}{2}\, dC \wedge d\bar{z} + \tfrac{1}{2}\, d\bar{C} \wedge dz.$$

Write

$$dC = C_0\omega + C_1\, dz + C_{\bar{1}}\, d\bar{z}. \tag{0.5}$$

Thus

$$d\omega = \left(i + \tfrac{1}{2}C_1 - \tfrac{1}{2}\,\overline{C_1}\right) dz \wedge d\bar{z} \quad (\operatorname{mod} \omega);$$

so

$$d\left(\left(1 - \frac{i}{2}C_1 + \frac{i}{2}\,\overline{C_1}\right)^{-1} \omega\right) = i\, dz \wedge d\bar{z} \quad (\operatorname{mod} \omega).$$

Let $(1 - \frac{i}{2}C_1 + \frac{i}{2}\overline{C_1})^{-1}\omega$ be our new ω and again let $dz = \omega_1$. Thus we have achieved

$$(0.6) \qquad d\omega = i\omega_1 \wedge \bar{\omega}_1 \pmod{\omega}$$

by using ω of the form

$$(0.7) \qquad \omega = \tfrac{1}{2}\left(du + iz\,d\bar{z} - i\bar{z}\,dz + \alpha\,du + \beta\,dz + \bar{\beta}\,d\bar{z}\right)$$

with α and β both $\mathcal{O}(2)$. (In fact, $\beta = \mathcal{O}(3)$. However, we will not need to use this extra control on β so we just take $\beta = \mathcal{O}(2)$.) Note that (0.6) can be written as

$$(0.8) \qquad d\omega = i\omega_1 \wedge \bar{\omega}_1 + b\omega \wedge \omega_1 + \bar{b}\omega \wedge \bar{\omega}_1$$

with $b = \mathcal{O}(1)$ and all first and second order derivatives bounded.

We have just seen that on a strictly pseudo-convex hypersurface in $\mathbf{C}^2$ it is always possible to choose ω and ω_1 such that $d\omega = i\omega_1 \wedge \bar{\omega}_1 \pmod{\omega}$. This choice is not unique. Cartan [2] has shown how to use any one choice to calculate quantities which are in fact independent of the choice. Consider, for instance, a set of equations

$$(0.9) \qquad \omega_1 = \mu\omega, \qquad d\mu = F(x, \mu)\omega,$$

where F is smooth on $M \times \mathbf{C}$. Along any curve in M which is transverse to H there is uniquely defined a function μ such that the first equation is valid when evaluated on any tangent vector to the curve. The second equation then either is or is not valid when evaluated at tangents to the curve and for that unique function μ. When equations (0.9) are written in terms of local coordinates, we have a second order equation for the curve. Through each point x and in each direction transverse to H there is a unique (unparametrized) curve which satisfies (0.9). From the work of Cartan [2] we know that the curves so defined from the following system are independent of our choice of ω and ω_1 and so define a CR invariant system of curves:

$$(0.10) \qquad \omega_1 = -\mu\omega, \qquad d\mu = \left(i\mu|\mu|^2 + \tfrac{1}{2}ic\mu - \tfrac{1}{6}\bar{l} - \bar{b}|\mu|^2\right)\omega.$$

Here b is given by (0.8). Then c and l are given by $c = \bar{b}_1$ and $l = c_1 - bc - 2ib_0$, where the subscripts come from the convention (0.5). Thus the second equation in (0.10) may be rewritten for $|\mu| \neq 0$ as

$$(0.11) \qquad d\mu = \left(i\mu|\mu|^2 + |\mu|^2 B(z, u, \mu)\right)\omega,$$

where B is bounded for $|\mu| > \varepsilon > 0$ as is any derivative with respect to z, $\bar{z}$ or u. Further $|B_\mu|$ and $|B_{\bar{\mu}}|$ are both less than $C|\mu|^{-2}$ for $|\mu|$ large. (We will later only need assume that they are less than $C|\mu|^{-1}$.)

The curves satisfying (0.10) are called *chains*. These equations are sufficiently complicated that not many properties of the chains are known. Before stating our results along these lines, we would like to mention two previous results. In [1] it is shown that there exists a compact $M^3 \subset \mathbf{C}^2$ which is strictly pseudo-convex everywhere with two particular points p and q in M such that there is no chain which connects p to q. In [4] it is shown that chains can misbehave even in the neighborhood of a point. For in this paper Fefferman showed that the hypersurface $v = |z|^2 + 2u|z|^8$ has chains which spiral in to the origin. Along such a chain necessarily $|\mu| \to \infty$. In [4] the chains were represented as the light rays of a metric on a circle bundle over M. Thus they are defined by a second order system of equations on a four-dimensional fibre bundle over M with compact fibre. In §1 we include a discussion, based directly on (0.10), that shows that chains can be defined by a first order system of equations on a five-dimensional fibre bundle over M with compact fibres. Further in each fibre we find a closed orbit. Any orbit which approaches this closed orbit corresponds to a chain on M which spirals in to a point. This provides a qualitative "explanation" for Fefferman's spirals. Because of the difficulty in studying any particular example, it is, however, not clear that this "explanation" in fact contributes real information.

The results of [1] and [4] show that chains behave quite differently than that prototype of geometrically defined curves, the geodesics of Riemannian geometry. Our purpose in this paper is to show that the chains do behave like geodesics in at least one basic regard.

Theorem. *Let $M^3 \subset \mathbf{C}^2$ be strictly pseudo-convex at p. There is an open neighborhood U of p in M such that for each $q \in U$ there is a smooth chain which passes through both p and q.*

The basic idea of our proof is simple. We work in coordinates as in (0.7). Consider the two sets

$$S(R) = \left\{(z,u) \colon |u| \leqslant \frac{1}{R}|z|, |z| \leqslant \delta(R)\right\},$$

$$T(R) = \left\{(z,u) \colon |u| \geqslant \frac{1}{R}|z|, |z| + |u| \leqslant \delta(R)\right\}.$$

Here R can be any large number and $\delta(R)$ monotonically decreases to zero. Near the origin, $S(R)$ is a neighborhood of the z-plane less the origin, $T(R)$ is the complement of such a neighborhood, and $S \cup T$ contains an open neighborhood of the origin. It is to be expected that a chain which connects the origin to a point p in S would start in a direction very close to H. Thus $|\mu|$ would initially be large and the equations (0.10) would be close to those of the hyperquadric. Any two points on the hyperquadric can be connected by a chain, so we may expect that we can find a chain from 0 to p by asymptotically comparing solutions to (0.10) and the corresponding equations for Q as $|\mu| \to \infty$. This is carried out in §2.

On the other hand, we may expect that a chain which connects the origin to a point q in T would start in a direction far from H. But away from H the equations have no singularities and we just need investigate when an equation of the form

$$z'' = F(z, z', t), \qquad z(0) = 0,\ z(1) = z_1,$$

has a solution for a given z_1. This is done in §3.

Finally in §4 the theorem is shown to hold in higher dimension and also for CR structures which do not come from real hypersurfaces in some $\mathbf{C}^N$. Thus §4 starts with the definition of an abstract CR structure and then discusses how the proofs in §§2 and 3 must be modified to yield the more general result. Incidentally, chains are defined for all nondegenerate CR hypersurfaces rather than just for strictly pseudo-convex hypersurfaces. However it can be seen that without strict pseudo-convexity there may exist nearby points which do not lie on a common chain. (See the discussion following the Theorem in §4.)

1. Let us start with a discussion of the chains on the hyperquadric Q. The equations of the chains are obtained from (0.10) by setting $b \equiv 0$. Thus we have

$$\omega_1 = -\mu\omega, \qquad d\mu = i|\mu|^2\mu\omega \tag{1.1}$$

with $\omega_1 = dz$ and $\omega = \frac{1}{2}(du + iz\,d\bar{z} - i\bar{z}\,dz)$. Since

$$d|\mu|^2 = \mu\,d\bar{\mu} + \bar{\mu}\,d\mu = -i|\mu|^4 + i|\mu|^4 = 0$$

we see that $|\mu|$ is a constant along each chain. It is convenient to choose a time parameter such that, along a given chain,

$$\omega = \frac{1}{|\mu|^2}, \quad \omega_1 = -1/\bar{\mu}, \quad d\mu = i\mu. \tag{1.2}$$

That is, we consider the system

$$\dot{u} = \frac{2}{|\mu|^2} - iz\dot{\bar{z}} + i\bar{z}\dot{z}, \quad \dot{z} = -1/\bar{\mu}, \quad \dot{\mu} = i\mu. \tag{1.3}$$

In doing this we lose the solution given by the u-axis since here $\mu = 0$. We add the initial conditions

$$u(0) = 0, \quad z(0) = 0, \quad \mu(0) = \nu. \tag{1.4}$$

The unique solution to the initial value problems (1.3), (1.4) is

$$z(t) = \frac{i}{\bar{\nu}}(e^{it} - 1), \quad u(t) = \frac{2\sin t}{|\nu|^2}, \quad \mu(t) = \nu e^{it}. \tag{1.5}$$

From this it is easy to see that the chains through the origin cover all of Q. Since the u-axis is a chain we need to show that every point off the u-axis also lies on a chain through the origin.

Lemma 1.1. *For any (z^*, u^*) with $z^* \neq 0$ there is some ν^* and some t^* such that the solution to* (1.3), (1.4) *evaluated at $\nu = \nu^*$ and $t = t^*$ gives (z^*, u^*). We may take $|t^*| \leqslant \pi$. Further if (z^*, u^*) satisfies*

$$|u^*| \leqslant \frac{1}{R}|z^*|, \quad |z^*| \leqslant \frac{1}{R}, \tag{1.6}$$

then $|\nu^*| \geqslant \sqrt{2}\,R$.

We omit the simple proof.

It follows that if $|\nu| = R$, then the solution to (1.3), (1.4) can never satisfy both inequalities (1.6). We will need a quantitative version of this. We claim there is a positive constant ε_0 such that for no t with $0 < |t| \leqslant 3\pi/2$ and no $R > 0$ does the solution to (1.3), (1.4) satisfy

$$|u(\nu, t)| \leqslant \frac{1}{R}|z| + \frac{\varepsilon_0 |t|}{R^2}, \quad |z(\nu, t)| \leqslant \frac{1 + \varepsilon_0}{R} \tag{1.7}$$

when $|\nu| = R$.

Now let

$$X(\nu, t) = \begin{pmatrix} z \\ \bar{z} \\ u \end{pmatrix}$$

be part of the solution (1.3), (1.4). Let $D = \det(X_\nu, X_{\bar{\nu}}, X_t)$. Thus D is the Jacobian determinant of the transformation $(\nu, t) \to (z(\nu, t), u(\nu, t))$ given by the solution to (1.3), (1.4). A simple computation yields

$$D = 4(1 - \cos t)/|\nu|^6. \tag{1.8}$$

There is a very nice geometric interpretation of chains on the hyperquadric which among other things provides another proof that any two points on the hyperquadric can be connected by a unique chain. Namely, the chains on the hyperquadric Q are the intersections of Q with complex lines. We may simplify this further by using a projective biholomorphism on $\mathbf{C}^2$ to take Q to the unit sphere S. Then the chains on S are again the intersections with complex lines and so are circles. As a complex line through $p \in S$ approaches the complex tangent line, the circles become smaller and so their curvatures approach infinity.

To see that the chains on Q are the intersections with complex lines, we return to the defining function $r = (z_2 - \bar{z}_2)/2i - |z_1|^2$. Then

$$dz_2 = 2i\partial r + 2i\bar{z}_1 dz_1 = 2\omega + 2i\bar{z}_1 dz_1$$

on Q. Now on the complex line $\alpha z_1 + \beta z_2 = \gamma$ we have

$$\alpha dz_1 = -\beta dz_2,$$

so on the intersection of this line and Q we have

$$\alpha dz_1 = -\beta(2i\partial r + 2i\bar{z}_1 dz_1) = -\beta(2\omega + 2i\bar{z}_1 dz_1).$$

Setting $\mu = 2\beta/(\alpha + 2i\beta\bar{z}_1)$ we have $dz_1 = -\mu\omega$. A simple computation shows that $d\mu = i\mu|\mu|^2\omega$ and so we are done.

If one could show that the chains on M were the intersections of M with a natural family of surfaces, then, perhaps, one could dispense with the methods of this paper and simply construct the chain connecting two given points. However on the one hand simple examples show that a chain on a C^∞ hypersurface need not lie even on the boundary of a Riemann surface while on the other it does not seem useful to know that every chain on a real analytic hypersurface is real analytic and therefore lies on its own complexification.

The equations for chains (0.10) have a singularity as μ goes to infinity. This is seen clearly on S^3 since as the circles shrink their curvatures go to infinity. However, there is a simple way to "compactify" these equations for a general CR structure in order to end up with a flow without singularities. This observation will be used in §2 but not in an essential way. However, it is important for motivation and is of some interest in itself, so we present it here.

We start with a CR manifold M^3. Let P be the projectivized tangent bundle of M. The fibre of P above any point p in M is the set of unoriented directions at p. In the presence of a metric, P can also be thought of as the unit sphere bundle over M with antipodal points identified. Let $P_H \subset P$ consist of those directions which belong to the distinguished 2-plane distribution H. Thus the fibre of P_H is some projective line P^1 in the projective plane P^2 which is the fibre of P. We consider P_H as giving the line at infinity in each fibre. Note that an equation $\omega_1 + \alpha(x)\omega = 0$ determines a section of P which never lies in P_H. In particular the equation for chains (0.10) can be thought of as defining a direction field on $P - P_H$. After any normalization this may be thought of as a vector field on $P - P_H$. Now, when M has a metric, the equation for geodesics gives a tangent vector field on the unit sphere bundle. Thus geodesics on M come from a flow on a bundle with compact fibres while chains on M come from a flow on a bundle with noncompact fibres. We will now see how to compactify the latter by defining a flow on all of P which extends the flow for chains.

We cover P by three patches of homogeneous coordinates in the following manner. First fix some coordinates $x = (x_1, x_2, x_3)$ for M^3. Let $\mu_1 = \xi + i\eta$ with ξ and η unrestricted, $\mu_2 = u + iv$ with $|v| < 1$, and $\mu_3 = r + is$ with $|s| < 1$. Define $X_j \in TM_x^3$ by

$$\begin{aligned} \omega(X_1) &= 1, \quad \omega_1(X_1) = -\mu, \\ \omega(X_2) &= v, \quad \omega_1(X_2) = -(1 + iu), \\ \omega(X_3) &= s, \quad \omega_1(X_3) = -(r + i). \end{aligned}$$

Finally let $p_j \in P$ be the line in TM_x^3 generated by X_j. Our coordinate patches are

$$\begin{aligned} P_1 &= \{(x, p_1(\mu_1)) \colon \mu_1 \in \mathbf{C}\}, \\ P_2 &= \{(x, p_2(\mu_2)) \colon |\operatorname{Im} \mu_2| < 1\}, \\ P_3 &= \{(x, p_3(\mu_3)) \colon |\operatorname{Im} \mu_3| < 1\}. \end{aligned}$$

It is clear that

$$P = P_1 \cup P_2 \cup P_3. \tag{1.9}$$

We will need the transition functions between our coordinate patches. So again let $p \in P$ represent the line generated by $X \in TM_x$. Let $\omega(X) = a$ and $\omega_1(X) = b + ic$. Then $p \in P_1 \cap P_2$ if $a \neq 0$, $b \neq 0$. So

$$\mu = -(b + ic)/a, \qquad u + iv = c/b - ia/b$$

and thus

$$\mu = (1 + iu)/v \tag{1.10}$$

gives the transition function from P_2 to P_1. Similarly, the transition function from P_3 to P_1 is

$$\mu = (r + i)/s \tag{1.11}$$

and the transition function from P_3 to P_2 is

$$u + iv = (1 + is)/r. \tag{1.12}$$

Note for later use that points in P_1 with $|\mu| < 1$ do not belong to $P_2 \cup P_3$.

We now show that the curves which satisfy (0.10) are orbits of a nonsingular vector field on P. We in fact prove a more general result which clearly contains this.

We consider equations of the form

$$\omega_1 + \mu\omega = 0, \qquad d\mu + F(x, \mu)\omega = 0, \tag{1.13}$$

where we assume that $y^3 F(x, \alpha y^{-1})$ is the restriction of some smooth function $H(x, \alpha, y)$ for $\alpha \in \mathbf{C}$, $y \in \mathbf{R}$ and that this H satisfies

$$(\operatorname{Im}\alpha)(\operatorname{Re} H(x, \alpha, 0)) - (\operatorname{Re}\alpha)(\operatorname{Im} H(x, \alpha, 0)) \neq 0$$

when $\alpha \neq 0$.

Proposition. *There is a smooth nowhere zero vector field on P whose orbits in* P_1 *satisfy* (1.13).

Proof. We will essentially eliminate the singularity at infinity by our choice of a time variable. The natural choice which we have already used in (1.2) will introduce a singularity at $\mu = 0$. This is avoided by use of a cut-off function. So let $\phi(r)$ be a C^∞ function supported in $\{r\colon |r| \leqslant 1\}$ with $0 \leqslant \phi \leqslant 1$ and $\phi \equiv 1$ near $r = 0$. Set $\psi(r) = 1 - \phi + r^2\phi$. In each coordinate patch P_i we define a vector field V_i as follows. In P_1 let V_1 satisfy

$$\omega(V_1) = \frac{\psi(|\mu|)}{|\mu|^2}, \quad \omega_1(V_1) = -\frac{\psi(|\mu|)}{\bar{\mu}},$$
$$d\mu(V_1) = -\frac{F(x,\mu)\psi(|\mu|)}{|\mu|^2}. \tag{1.14}$$

In P_2 let V_2 satisfy

$$\omega(V_2) = \frac{v^2}{1+u^2}, \qquad \omega_1(V_2) = \frac{-v}{1-iu},$$
$$(-(1+iu)\,dv + iv\,du)(V_2) = -v^4(1+u^2)^{-1}F\big(x,(1+iu)v^{-1}\big). \tag{1.15}$$

In P_3 let V_3 satisfy

$$\omega(V_3) = \frac{s^2}{1+r^2}, \qquad \omega_1(V_3) = \frac{-s}{r-i},$$
$$(-(i+r)\,ds + s\,dr)(V_3) = -s^4(r^2+1)^{-1}F\big(x,(r+i)s^{-1}\big). \tag{1.16}$$

V_1 is uniquely determined, smooth, and nowhere zero in P_1. Let us show that the same is true for V_2 in P_2 and that $V_1 = V_2$ on $P_1 \cap P_2$. If we write

$$V_2 = \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} + \beta_1 \frac{\partial}{\partial u} + \beta_2 \frac{\partial}{\partial v},$$

then α_i is determined by the first two equations in (1.15). Rewrite the last equation in (1.15) as the two equations

$$\beta_2 = v(1+u^2)^{-1}\operatorname{Re} H(x, 1+iu, v),$$
$$v\beta_1 = u\beta_2 - v(1+u^2)^{-1}\operatorname{Im} H(x, 1+iu, v).$$

V_2 is clearly smooth and nonzero where $v \neq 0$ while at $v = 0$ we have $\beta_2 = 0$ and

$$\beta_1 = (1+u^2)^{-1}\{u\operatorname{Re} H(x,1+iu,0) - \operatorname{Im} H(x,1+iu,0)\}.$$

So $\beta_1 \neq 0$. Hence V_2 is uniquely determined, smooth, and nowhere zero in P_2. The analogous result holds for V_3 in P_3.

On $P_1 \cap P_2$ we have, from (1.10),

$$d\mu = iv^{-1}\,du - (1 + iu)v^{-2}\,dv,$$

$$|\mu|^2 = (1 + u^2)v^{-2}.$$

Thus, since $\psi \equiv 1$ on $P_1 \cap P_2$,

$$\omega(V_2) = \frac{v^2}{1 + u^2} = \frac{1}{|\mu|^2} = \omega(V_1),$$

$$\omega_1(V_2) = \frac{-v}{1 - iu} = -\frac{1}{\bar{\mu}} = \omega_1(V_1),$$

$$d\mu(V_2) = \frac{1}{v^2}(iv\,du - (1 + iu)\,dv) = \frac{1}{v^2}\frac{(-v^4)}{(1 + u^2)^{-1}}F(x, \mu)$$

$$= -\frac{1}{|\mu|^2}F(x, \mu) = d\mu(V_1).$$

Hence $V_1 = V_2$ on $P_1 \cap P_2$. The same type of reasoning also shows $V_1 = V_3$ on $P_1 \cap P_3$. Finally in showing that $V_2 = V_3$ on $P_2 \cap P_3$ one makes use of the fact that

$$u^3 H\left(x, \frac{1}{u} + i, 0\right) = H(x, 1 + iu, 0).$$

This fact follows immediately from the definition of H.

2. In this section we study chains which have an initial direction close to H. We show that any such chain remains sufficiently close to the chain in Q with the same initial direction. Let us start by writing down the two initial value problems. We will use $X = (z, u)$ and μ for the solution on Q and $Y = (\zeta, \eta)$ and σ for the solution in the given CR structure. We introduce a parameter t into the equations for the chains (0.10) by taking $\omega(\partial/\partial t) = 1/|\mu|^2$ in the first case and $\omega(\partial/\partial t) = 1/|\sigma|^2$ in the second. We obtain for Q

$$\dot{u} = \frac{2}{|\mu|^2} - iz\dot{\bar{z}} + i\bar{z}\dot{z}, \quad \dot{z} = -\frac{1}{\bar{\mu}}, \quad \dot{\mu} = i\mu, \tag{2.1}$$

$$X(0) = 0, \qquad \mu(0) = \nu.$$

And for our general CR structure we have

$$\dot{\eta} = \frac{2}{|\sigma|^2} - i\zeta\dot{\bar{\zeta}} + i\bar{\zeta}\dot{\zeta} + \alpha(\zeta, \eta)\dot{\eta} + \beta(\zeta, \eta)\dot{\zeta} + \bar{\beta}(\zeta, \eta)\dot{\bar{\zeta}},$$

$$\dot{\zeta} = -\frac{1}{\bar{\sigma}}, \qquad \dot{\sigma} = i\sigma + B(\zeta, \eta, \sigma), \tag{2.2}$$

$$Y(0) = 0, \qquad \sigma(0) = \nu.$$

Let us use as an abbreviation

$$\|f_\nu(\nu,t)\| = \tfrac{1}{2}\{|f_\nu(\nu,t)| + |f_{\bar\nu}(\nu,t)|\}$$

when f is a complex valued function. Note that if f is real valued, then $\|f_\nu\| = |f_\nu|$. Then using the estimates after (0.7) and (0.11) we see that we can assume the coefficients in (2.2) satisfy: There exist constants ε^* and C^* such that if $|Y| < \varepsilon^*$ and $|\sigma| \geqslant C^*$, then

$$|\alpha(\zeta,\eta)| + |\beta(\zeta,\eta)| \leqslant C^*\left(|\zeta|^2 + |\eta|^2\right), \tag{2.3}$$

$$\begin{aligned} &\|\alpha_\zeta(\zeta,\eta)\| + \|\beta_\zeta(\zeta,\eta)\| \leqslant C^*(|\zeta| + |\eta|),\\ &|\alpha_\eta(\zeta,\eta)| + |\beta_\eta(\zeta,\eta)| \leqslant C^*(|\zeta| + |\eta|), \end{aligned} \tag{2.4}$$

$$|B(\zeta,\eta,\sigma)| \leqslant C^*, \tag{2.5}$$

$$\|B_\zeta(\zeta,\eta,\sigma)\| + |B_\eta(\zeta,\eta,\sigma)| \leqslant C^*, \tag{2.6}$$

$$\|B_\sigma(\zeta,\eta,\sigma)\| \leqslant C^*/|\sigma|. \tag{2.7}$$

Temporarily, let

$$f_n = |\nu|^n|\zeta| + |\nu|^{n+1}\|\zeta_\nu\| + |\nu|^{n+1}|\eta| + |\nu|^{n+2}\|\eta_\nu\|.$$

We will be able to concisely state our estimates by using the following two function spaces. (The variable t always satisfies $|t| \leqslant 3\pi/2$.)

$$\mathscr{A}_n = \Big\{Y = (\zeta(\nu,t),\ \eta(\nu,t)):\ \limsup_{|\nu|\to\infty} f_n < C,$$
$$\text{where } C \text{ depends on } Y \text{ but not on } t\Big\},$$

$$t\mathscr{A}_n = \{Y = (\zeta(\nu,t),\ \eta(\nu,t)):\ Y = t\tilde{Y},\ \tilde{Y} \in \mathscr{A}_n\}.$$

Our basic result in this section is the next lemma which compares the solutions to (2.1) and (2.2).

Lemma 2.1. $\dot{Y} - \dot{X} \in t\mathscr{A}_2$.

The proof is broken down into several steps.

Lemma 2.2. *If $|\nu|$ is large enough, then the solution to* (2.2) *exists on the interval $|t| \leqslant 3\pi/2$ and there satisfies*

$$|Y(t)| \leqslant \varepsilon^*, \qquad |\sigma(t)| \geqslant |\nu|/2. \tag{2.8}$$

The proof is based on the observation that if $|Y(\nu,t)| \leqslant \varepsilon^*$ and $|\sigma(t)| \geqslant |\nu|/2$ on $|t| \leqslant t^*$ with $t^* < 3\pi/2$ and if $|\nu|$ is large enough, then indeed one has $|Y(\nu,t^*)| \leqslant \varepsilon^*/2$ and $|\sigma(\nu,t^*)| \geqslant 2|\nu|/3$. Thus (2.8) must hold on $|t| \leqslant 3\pi/2$. (The details of a similar proof are given for the next lemma.)

Henceforth we only consider solutions $(Y(\nu, t),\ \sigma(\nu, t))$ with $|\nu|$ sufficiently large and we restrict t by $|t| \leqslant 3\pi/2$. The estimates (2.3)–(2.7) are valid for such solutions.

Lemma 2.3. (a) $\|(\partial\sigma/\partial\nu)(t)\| \leqslant 2$.

(b) $\partial Y/\partial t \in \mathscr{A}_1$.

(c) $Y \in t\mathscr{A}_1$.

Proof. It follows from the previous lemma, equations (2.2), and inequality (2.3) that in the interval $[0, 3\pi/2]$

$$(2.9) \qquad |\dot{\zeta}| \leqslant 2|\nu|^{-1}, \quad |\zeta| \leqslant 2t|\nu|^{-1}, \quad |\dot{\eta}| \leqslant c|\nu|^{-2}, \quad |\eta| \leqslant ct|\nu|^{-2}.$$

Let $t^* \in [0, 3\pi/2]$ be the first time if it exists that one of the inequalities

$$(2.10) \qquad \left\| \frac{\partial\sigma}{\partial\nu}(t) \right\| < 2, \quad \|\zeta_\nu(t)\| < 1, \quad \|\eta_\nu(t)\| < 1$$

is violated. Otherwise set $t^* = 3\pi/2$. We show that for $0 \leqslant t < t^*$ and $|\nu|$ large $\|(\partial\sigma/\partial\nu)(t)\|$, $\|\zeta_\nu(t)\|$ and $\|\eta_\nu(t)\|$ are bounded away from 2, 1 and 1 respectively. This would show that the inequalities (2.10) are valid in the interval $0 \leqslant t \leqslant 3\pi/2$. Essentially the same argument of course works for t negative.

We have

$$|\zeta_\nu(t)| = |\bar{\sigma}_\nu/\bar{\sigma}^2| \leqslant 8\|\sigma_\nu\|/|\nu|^2 \leqslant 16/|\nu|^2$$

on $0 \leqslant t \leqslant t^*$; and we have the same estimate for $|\dot{\zeta}_{\bar{\nu}}(t)|$. Thus $\|\dot{\zeta}_\nu(t)\| \leqslant 16/|\nu|^2$ and, since $\zeta_\nu(0) = 0$,

$$\|\zeta_\nu(t)\| \leqslant 16t/|\nu|^2.$$

Similarly

$$\|\dot{\eta}_\nu\| \leqslant C/|\nu|^3, \quad \|\eta_\nu(t)\| \leqslant Ct/|\nu|^3.$$

Finally, if we start with the equation $(d/dt)(e^{-it}\sigma) = B(\zeta, \eta, \sigma)$ and use the above estimates for $\|\zeta_\nu\|$ and $\|\eta_\nu\|$ together with the conditions (2.6) and (2.7) and impose a further largeness condition on $|\nu|$, we obtain

$$\|\sigma_\nu\| \leqslant \frac{3}{2}\left(1 + \frac{C}{|\nu|^2}\right).$$

Thus for $|\nu|$ large

$$\|\zeta_\nu(t)\| \leqslant 1/2, \quad \|\eta_\nu(t)\| \leqslant 1/2, \quad \|\sigma_\nu(t)\| \leqslant 7/4$$

when $0 \leqslant t < t^*$. The definition of t^* then shows that $t^* = 3\pi/2$. But we had already seen that

$$|\zeta(t)| \leqslant 2t/|\nu|, \quad |\eta(t)| \leqslant Ct/|\nu|^2$$

and now that

$$\|\zeta_\nu(t)\| \leqslant 16t/|\nu|^2, \quad \|\eta_\nu(t)\| \leqslant Ct/|\nu|^3.$$

Thus $Y \in t\mathscr{A}_1$. Further we have $|\dot\zeta| \leqslant 2|\nu|^{-1}$, $|\dot\eta| \leqslant C|\nu|^{-2}$ and $\|\dot\zeta_\nu\| \leqslant 16|\nu|^{-2}$, $\|\dot\eta_\nu\| \leqslant C|\nu|^{-3}$. So $\partial Y/\partial t \in \mathscr{A}_1$. This completes the proof of Lemma 2.3.

Next we need to compare the μ and σ components of the solutions to (2.1) and (2.2).

Lemma 2.4. *There is some constant C such that*

$$|\sigma(t) - \mu(t)| \leqslant C|t| \quad \textit{and} \quad \|\sigma_\nu(t) - \mu_\nu(t)\| \leqslant C|t||\nu|.$$

Proof. From Lemma 2.2 we know that the estimates (2.5) and (2.6) are valid at $(Y(\nu,t), \sigma(\nu,t))$ everywhere in our interval $|t| \leqslant 3\pi/2$. Write $\sigma = \mu + g$. Then g is the solution to

$$\dot g = ig + B(Y(\nu,t), \sigma(\nu,t)), \qquad g(0) = 0$$

and by considering $(d/dt)e^{-it}g$ we easily derive the estimates

$$|g(t)| \leqslant C|t| \quad \text{and} \quad \|g_\nu\| \leqslant C|t|/|\nu|,$$

thus proving the lemma.

Note the following consequences of this lemma:

(2.11) $$|\bar\sigma^{-1} - \bar\mu^{-1}| \leqslant Ct|\nu|^{-2},$$

(2.12) $$|(\bar\sigma^{-1} - \bar\mu^{-1})_\nu| \leqslant Ct|\nu|^{-3},$$

(2.13) $$\big||\sigma|^{-2} - |\mu|^{-2}\big| \leqslant Ct|\nu|^{-3},$$

(2.14) $$\left|\left(|\sigma|^{-2} - |\mu|^{-2}\right)_\nu\right| \leqslant Ct|\nu|^{-4}.$$

We are now able to prove Lemma 2.1. Comparing the solutions to (2.1) and (2.2), we have

(2.15) $$|\dot\zeta - \dot z| = |-1/\bar\sigma + 1/\bar\mu| \leqslant Ct/|\nu|^2$$

and

(2.16) $$\|\dot\zeta_\nu - \dot z_\nu\| = \|(-1/\bar\sigma + 1/\bar\mu)_\nu\| \leqslant Ct/|\nu|^3.$$

Of course, we then also have

$$|\zeta - z| \leqslant Ct^2/|\nu|^2, \quad \|\zeta_\nu - z_\nu\| \leqslant Ct^2/|\nu|^3.$$

Next we need estimates for $|\eta - u|$. Comparing the equations (2.1) and (2.2), we see that

$$\tfrac{1}{2}|\dot\eta - \dot u| \leqslant \left|1/|\sigma|^2 - 1/|\mu|^2\right| + |\zeta|\,|\dot z - \dot\zeta| + |\dot z|\,|\zeta - z| + |\alpha|\,|\dot\eta| + |\beta|\,|\dot\zeta|.$$

We now use the inequality (2.13), Lemma 2.3, (2.15) and (2.3) to obtain

$$|\dot{\eta} - \dot{u}| \leqslant Ct/|\nu|^3. \tag{2.17}$$

Finally we want estimates for $|(\dot{\eta} - \dot{u})_\nu|$. One term will involve α_ν. So note

$$|\alpha_\nu| = \left|\alpha_\zeta \zeta_\nu + \alpha_{\bar\zeta}\bar\zeta_\nu + \alpha_\eta \eta_\nu\right| \leqslant Ct^2/|\nu|^3$$

and in the same way we establish

$$\|\alpha_\nu\| \leqslant Ct^2/|\nu|^3 \quad \text{and} \quad \|\beta_\nu\| \leqslant Ct^2/|\nu|^3.$$

We have

$$\begin{aligned}\tfrac{1}{2}\|(\dot{\eta} - \dot{u})_\nu\| \leqslant{} & \left\|\left(1/|\sigma|^2 - 1/|\mu|^2\right)_\nu\right\| + 2\|\zeta_\nu\|\,|\dot{z} - \dot{\zeta}| \\ & + 2|\zeta|\,\|(\dot{z} - \dot{\zeta})_\nu\| + 2\|\dot{z}_\nu\|\,|\zeta - z| + 2|\dot{z}|\,\|(z - \zeta)_\nu\| \\ & + \|\alpha_\nu\|\,|\dot{\eta}| + |\alpha|\,\|\dot{\eta}_\nu\| + 2\|\beta_\nu\|\,|\dot{\zeta}| + 2|\beta|\,\|\dot{\zeta}_\nu\|.\end{aligned}$$

Now we use (2.14), Lemma 2.3, (2.15), (2.16) and the above estimates on $\|\alpha_\nu\|$ and $\|\beta_\nu\|$ to obtain

$$\|(\dot{\eta} - \dot{u})_\nu\| \leqslant Ct/|\nu|^4. \tag{2.18}$$

Now (2.15), (2.16), (2.17) and (2.18) are precisely the assertion that $\dot{Y} - \dot{X} \in t\mathscr{A}_2$. This concludes the proof of Lemma 2.1.

We now show that this lemma provides us with a transversality result.

Let D denote the determinant

$$\begin{vmatrix} \zeta_\nu & \zeta_{\bar\nu} & \zeta_t \\ \bar\zeta_\nu & \bar\zeta_{\bar\nu} & \bar\zeta_t \\ \eta_\nu & \eta_{\bar\nu} & \eta_t \end{vmatrix}.$$

It follows easily from Lemma 2.1 and (1.8) that

$$D \geqslant 4(1 - \cos t)/|\nu|^6 - Ct^3/|\nu|^7 \tag{2.19}$$

for some C. Thus there is some large R such that $D > 0$ provided $|\nu| \geqslant R$ and $0 < |t| \leqslant 3\pi/2$. Note that the estimates of Lemma 2.1 are precisely of the right form in ν to yield this positivity result. (It is not clear whether the determinant is in fact positive for all ν. The corresponding question for Riemannian geometry depends on the presence of conjugate points.)

It is convenient to divide the set of points near the origin into two (nondisjoint) subsets: Let $(z(\nu, t),\ u(\nu, t))$ be the solution to (2.1). This solution is given explicitly by (1.5). Then each point (z^*, u^*) belongs to at least one of the sets $\{(z(\nu, t),\ u(\nu, t)):\ \nu \in \mathbf{C},\ 0 \leqslant t \leqslant \pi\}$ and $\{(z(\nu, t),\ u(\nu, t)):\ \nu \in \mathbf{C},\ -\pi \leqslant t \leqslant 0\}$. We shall consider only the first set and show that each of its points can be connected to the origin by a chain. The same argument, mutatis mutandis, works for the second set.

Thus for each point (z^*, u^*) we consider, there are unique values of ν^* and t^*, with $0 \leqslant t^* \leqslant \pi$, for which $z(\nu^*, t^*) = z^*$ and $u(\nu^*, t^*) = u^*$. Let $T = t^* + \pi/4$.

Let $F = \{(z^*, u^*, \sigma): \sigma \in \mathbf{C}\}$ be the fibre above the given point (z^*, u^*). For some R define

$$N = \{(Y(\nu, t), \sigma(\nu, t)): (Y, \sigma) \text{ solves } (2.2), |\nu| \geqslant R \text{ and } 0 \leqslant t \leqslant T\}.$$

Thus N is the orbit of the piece near infinity of the fibre over the origin. Note that if R is so large that $D > 0$, then the tangent plane to F at a point of $F \cap N$, if there is such a point, and the tangent plane to N at that point together span the whole tangent space. We say that any intersection of F and N is *transverse*.

Recall the projectivized tangent bundle P described in §1. The variable σ should be thought of as parametrizing each fibre except for the line at infinity. Now adjoin this line at infinity to N and F. Then N and F are compact. We will show that if

$$(2.20) \qquad |u^*| \leqslant \frac{1}{R}|z^*|, \quad |z^*| \leqslant \frac{1}{R},$$

then N intersects F provided only that R is large enough. This would establish that all points satisfying (2.20) can be connected to the origin by means of a chain.

We start with two results which show that the boundary of N does not intersect F. Notice that the pieces of the boundary given by $|\nu| = \infty$ and by $t = 0$ both lie in the fibre over the origin and so do not intersect F (except in the trivial case where (z^*, u^*) is the origin). Thus we need to show that neither $\{Y(\nu, t): |\nu| = R, 0 \leqslant t \leqslant T\}$ nor $\{Y(\nu, T): |\nu| \geqslant R\}$ can contain (z^*, u^*).

Lemma 2.5. *Let* (z^*, u^*) *satisfy* (2.20) *with* R *sufficiently large and let* $|\nu| \geqslant R$. *If for some* t_1 *with* $|t_1| \leqslant 3\pi/2$ *one has* $Y(\nu, t_1) = (z^*, u^*)$, *then* $|\nu| > R$.

Proof. If $z^* = \zeta(\nu, t_1)$ and $u^* = \eta(\nu, t_1)$ for such a given ν and t_1, then we see by comparing the solutions of (2.1) and (2.2) that

$$z(\nu, t_1) = z^* + \mathcal{O}\left(t_1^2/|\nu|^2\right) = z^* + \mathcal{O}\left(t_1^2/R^2\right),$$

$$u(\nu, t_1) = u^* + \mathcal{O}\left(t_1^2/|\nu|^3\right) = u^* + \mathcal{O}\left(t_1^2/R^3\right).$$

Thus

$$|z(\nu, t_1)| \leqslant 1/R + \mathcal{O}\left(t_1^2/R^2\right),$$

$$|u(\nu, t_1)| \leqslant |z|/R + \mathcal{O}\left(t_1^2/R^3\right).$$

Hence when R is large (1.7) is valid and we cannot have $|\nu| = R$. But we have already assumed $|\nu| \geqslant R$ and so we conclude $|\nu| > R$.

Lemma 2.6. *Let* (z^*, u^*) *satisfy* (2.20) *with* R *sufficiently large. Then* $|Y(\nu, T) - (z^*, u^*)| > 0$ *for all* ν, $|\nu| \geqslant R$.

Proof. We have

$$|\zeta(\nu,T) - z^*| \geqslant |z(\nu,T) - z(\nu^*,t^*)| - |\zeta(\nu,T) - z(\nu,T)|$$
$$\geqslant \left| \frac{e^{iT}-1}{\bar\nu} - \frac{e^{it^*}-1}{\bar\nu^*} \right| - \frac{C_2}{|\nu|^2},$$
$$|\eta(\nu,T) - u^*| \geqslant |u(\nu,T) - u(\nu^*,t^*)| - |\eta(\nu,T) - u(\nu,T)|$$
$$\geqslant 2\left| \frac{\sin T}{|\nu|^2} - \frac{\sin t}{|\nu^*|^2} \right| - \frac{C}{|\nu|^3}.$$

Thus it is enough to show that at least one of the following estimates holds:

$$\left| \frac{e^{iT}-1}{\bar\nu} - \frac{e^{it^*}-1}{\bar\nu^*} \right| \geqslant \frac{C}{|\nu|} \quad \text{or} \quad \left| \frac{\sin T}{|\nu|^2} - \frac{\sin t^*}{|\nu^*|^2} \right| \geqslant \frac{C}{|\nu|^2} \tag{2.21}$$

with C positive and independent of ν and ν^*.

Let $\lambda = \bar\nu/\bar\nu^*$. It is enough to find $C > 0$ such that for each λ one has

$$|e^{iT} - 1 - \lambda(e^{it^*} - 1)| \geqslant C \quad \text{or} \quad |\sin T - |\lambda|^2 \sin t^*| \geqslant C. \tag{2.22}$$

Note that $|e^{iT} - 1| = (2(1 - \cos T))^{1/2} \geqslant (2(1 - 1/\sqrt{2}))^{1/2} = a_0$, that $e^{it} - 1$ and $\sin t$ vanish to first order at the origin, and that $e^{it} - 1 \neq 0$ for $0 < t < 3\pi/2$. We use the following result which can be proved by dividing it into several cases depending on the values of t and λ.

Lemma 2.7. *Let* a, b, f, g, *be bounded functions of* t *on some interval* $|t| \leqslant M$ *satisfying*

$$f(0) = g(0) = 0, \quad f'(0) \neq 0, \quad g'(0) \neq 0,$$
$$f(t) \neq 0 \quad \textit{for } 0 < |t| \leqslant M,$$
$$|a(t)| \geqslant a_0 > 0, \qquad \left| b(t) - \left|\frac{a(t)}{f(t)}\right|^2 g(t) \right| > C_1 > 0.$$

Then there is some $C > 0$ *such that*

$$|a - \lambda f| + |b - |\lambda|^2 g| > C \tag{2.23}$$

for all t *in* $|t| \leqslant M$ *and all* $\lambda \in \mathbf{C}$.

We apply Lemma 2.7 to prove Lemma 2.6 by setting $a = e^{iT}$, $b = \sin T$, $f = e^{it} - 1$, and $g = \sin t$ with $T = t^* + \pi/4$. In checking that the hypotheses of Lemma 2.7 all hold we make use of the inequality

$$\left|\sin\left(t + \frac{\pi}{4}\right) - \left(\frac{1 - \cos(t + \pi/4)}{1 - \cos t}\right)\sin t\right| > c > 0 \tag{2.24}$$

valid for $0 \leqslant t \leqslant 5\pi/4$.

Thus the alternative in (2.22) is satisfied and so Lemma 2.6 is established.

Lemmas 2.5 and 2.6 yield that F does not intersect the boundary of N. We can now use a simple deformation argument to show that $N \cap F$ is nonempty. First pick a smooth family of CR structures $(\omega^\lambda, \omega_1)$ with ω^0 given by (0.2) and ω^1 given by (0.7). This latter defines the given CR structure. For all λ we let $\omega_1 = dz$. We assume the normalization $d\omega^\lambda = i\omega_1\overline{\omega}_1 \pmod{\omega^\lambda}$ and further that the estimates (2.3)–(2.7) hold uniformly for $0 \leqslant \lambda \leqslant 1$. Let N^λ be N for the λ-CR structure. The results of this section apply to each N^λ. Again fix some (z^*, u^*) satisfying

$$|u^*| \leqslant \frac{1}{R}|z^*|, \quad |z^*| \leqslant \frac{1}{R}.$$

Then $N^0 \cap F$ is nonempty. Further $\{\text{bdy}\, N^\lambda\} \cap F$ is empty for all λ. Finally, if for some λ, $N^\lambda \cap F$ is nonempty, then N^λ and F intersect transversally at each point. It follows that $N^\lambda \cap F$ is nonempty for all λ. In particular $N^1 \cap F$ is nonempty. Then $Y(\nu_1, t_1) = (z^*, u^*)$ for some ν_1 and some t_1; so $Y(\nu, t)$ is a chain through the origin which also contains the point (z^*, u^*). Thus we have proved the following result.

Proposition. *There is some large number R_0 such that, for each $R \geqslant R_0$, each point in the set*

$$S(R) = \{(z^*, u^*): |u^*| \leqslant |z^*|/R, |z^*| \leqslant 1/R\}$$

can be connected to the origin by some chain.

3. In this section we study chains whose initial directions are bounded away from H. We start with the chain equations which we now write as

$$\omega_1 = -\mu\omega, \qquad d\mu = A(z, u, \mu)\omega, \tag{3.1}$$

where as before

$$\begin{aligned} \omega &= \tfrac{1}{2}(du + iz\,d\bar{z} - i\bar{z}\,dz + \alpha(z,u)\,du + \beta(z,u)\,dz + \bar{\beta}(z,u)\,dz), \\ \omega_1 &= dz. \end{aligned} \tag{3.2}$$

Now we only need much weaker information on the coefficients and so we assume that α and β are smooth on $\{(z,u)\colon |z|+|u|<\varepsilon\}$ and vanish at the origin and that A is smooth on $\{(z,u,\mu)\colon |z|+|u|<\varepsilon,\ \mu\in\mathbf{C}\}$.

In §2 we chose a parametrization so as to introduce a $1/|\mu|^2$ factor. Certainly we do not want to do that now. Rather, note that if a curve satisfies (3.1), then ω restricted to the curve is nonzero and so, at least near the origin, du is also nonzero. So we may use u as a time variable. Set $t=u$ and consider the equations

$$\begin{aligned}\dot z &= -\tfrac12\mu\big(1+iz\dot{\bar z}-i\bar z\dot z+\alpha(z,t)+\beta(z,t)\dot z+\bar\beta(z,t)\dot{\bar z}\big),\\ \dot\mu &= \tfrac12 A(z,u,\mu)\big(1+iz\dot{\bar z}-i\bar z\dot z+\alpha(z,t)+\beta(z,t)\dot z+\bar\beta(z,t)\dot{\bar z}\big).\end{aligned}\tag{3.3}$$

These equations are defined for (z,t) near the origin and all $\mu\in\mathbf{C}$. Thus we always require, for some ε_1, that $|z|+|t|<\varepsilon_1$.

We will again impose the initial conditions

$$z(\nu,0)=0,\quad u(\nu,0)=0,\quad \mu(\nu,0)=\nu.\tag{3.4}$$

Lemma 3.1. *There exist some positive function $\varepsilon(r)$ and smooth functions $F=F(z,\bar z,w,\bar w,t)$ and $G=G(z,\bar z,w,\bar w,t)$ defined on*

$$S=\{(z,w,t)\colon |z|+|t|<\varepsilon(|w|)\}$$

and a smooth function $h(\nu)$ defined on $\mathbf{C}$ such that whenever $z(t)$ solves

$$\ddot z=F(z,\dot z,t),\tag{3.5}$$

$$z(0)=0,\qquad \dot z(0)=h(\nu),\tag{3.6}$$

then $z(t)$ together with $\mu(t)=G(z,\dot z,t)$ solves the initial value problem (3.3), (3.4).

Note that we may assume that $\varepsilon(r)$ is monotone decreasing and that $\varepsilon(0)$ is small.

Proof. Solve the first equation in (3.3) for $\dot z$. Use the conjugate of the result to eliminate $\dot{\bar z}$ from this equation. The result is

$$\dot z=-\tfrac12\mu(1+\alpha)+f(z,\mu,t)\tag{3.7}$$

with

$$|f|\leqslant C_1|\mu|^2(|z|+|t|)\quad\text{and}\quad \|f_\mu\|\leqslant C_1|\mu|(|z|+|t|)\tag{3.8}$$

on the set where $|\mu|(|z|+|t|)<\varepsilon_2$ for some small ε_2 and some constant C_1. Note that ε_2 may be replaced by any smaller quantity.

We now solve (3.7) for μ.

Lemma 3.2. *Let $\varepsilon(r)=\frac18\varepsilon_2/(1+r)$. There is a smooth function $G(z,w,t)$ defined on*

$$S=\{(z,w,t)\colon |z|+|t|<\varepsilon(|w|)\}$$

such that $\mu = G(z, w, t)$ *solves*

$$w = -\tfrac{1}{2}\mu(1+\alpha) + f(z,\mu). \tag{3.9}$$

Proof. We consider z, w, and t as fixed and we seek μ which satisfies (3.9). Let us start with $\mu_0 = 0$. Now, if $|\mu_n|(|z| + |t|) < \varepsilon_2$ and $|z| + |t| < \varepsilon_1$, then $f_n = f(z, \mu_n)$ is defined and we may set

$$\mu_{n+1} = 2(f_n - w)(1+\alpha)^{-1}. \tag{3.10}$$

One can easily show that this iteration is well defined and that $\{\mu_n\}$ has a limit $\mu = G(z, w)$ which satisfies (3.9). This completes the proof of Lemma 3.2. We now continue with the proof of Lemma 3.1.

We have from (3.3) and Lemma 3.2 that $\dot{\mu} = g(z, \dot{z}, t)$ with $g(z, w, t)$ smooth on S. We differentiate equation (3.7) and then substitute $\mu = G(z, \dot{z}, t)$ and $\dot{\mu} = g(z, \dot{z}, t)$. The result is of the form

$$\ddot{z} = F(z, \bar{z}, \dot{z}, \dot{\bar{z}}, t)$$

with $F(z, w, t)$ smooth on S. Finally we define $h(\nu)$ by $\nu = G(0, h)$. (For this we need $|G_w(0)| \neq |G_{\bar{w}}(0)|$ and this is obvious from (3.9) after we replace μ by $G(z, w, t)$.) The converse, which is the conclusion of Lemma 3.1, then follows from the fundamental existence and uniqueness theorem of ordinary differential equations.

We now consider the boundary value problem

$$\ddot{z} = F(z, \dot{z}, t), \qquad z(0) = 0,\ z(t^*) = z^*, \tag{3.11}$$

when $F(z, w, t)$ is smooth on $S = \{(z, w, t){:}\ |z| + |t| \leqslant \varepsilon(|w|)\}$. Because of this smoothness we have

$$|F(z,w,t)| + \|F_z(z,w,t)\| + \|F_w(z,w,t)\| \leqslant C(|w|) \tag{3.12}$$

for some monotone increasing function $C(r)$.

Lemma 3.3. *For each R there exist some $\delta(R)$ such that* (3.11) *can be solved whenever $|z^*| \leqslant R|t^*|$ provided $|z^*| + |t^*| < \delta(R)$.*

Proof. We use a simple iteration. Let $z_0 = tz^*/t^*$ for $|t| \leqslant |t^*|$ and, as long as it makes sense, set

$$\begin{aligned} z_n = {}& \int_0^t \int_0^\tau F(z_{n-1}(s), \dot{z}_{n-1}(s), s)\, ds\, d\tau \\ & + \frac{t}{t^*}\left(z^* - \int_0^{t^*} \int_0^\tau F(z_{n-1}(s), \dot{z}_{n-1}(s), s)\, ds\, d\tau\right). \end{aligned} \tag{3.13}$$

It is clear that if $z_n(t)$ can always be defined and converges in C^1 to $z(t)$, then $z(t)$ is actually twice differentiable and satisfies (3.11). Let us assume $\delta(R) \leqslant \varepsilon(2R)$. Then

$$|z_0(t)| + |t| \leqslant |z^*| + |t^*| \leqslant \delta(R) \leqslant \varepsilon(2R)$$

while $|\dot{z}_0| = |z^*|/|t^*| \leqslant R$ and so

$$|z_0(t)| + |t| \leqslant \varepsilon(|\dot{z}_0(t)|)$$

for $|t| \leqslant |t^*|$.

We write $F_n(t) = F(z_n(t), \dot{z}_n(t), t)$. Thus F_0 is defined and hence so is z_1. Next assume z_n is defined and satisfies

$$|z_n(t)| \leqslant \varepsilon(2R), \quad |\dot{z}_n(t)| \leqslant 2R. \tag{3.14}$$

Then $|z_n(t)| + |t| \leqslant \varepsilon(|\dot{z}_n(t)|)$; hence F_n is defined and satisfies $|F_n(t)| \leqslant C(R)$. It is then easy to show that (3.14) also holds for $n + 1$ provided $|z^*| + |t^*| \leqslant \varepsilon_1(R)$ and $|z^*| \leqslant R|t^*|$. Thus (3.13) can be used to define a sequence $\{z_k(t)\}$.

Next we use (3.12) to write

$$|F_k(t) - F_{k-1}(t)| \leqslant C(2R)|||z_k - z_{k-1}|||,$$

where we use

$$|||z||| = \max_{|t| \leqslant |t^*|} \{|z(t)| + |\dot{z}(t)|\}.$$

Then, as is easily seen, $|||z_{k+1} - z_k||| \leqslant \frac{1}{2}|||z_k - z_{k-1}|||$ provided $|t^*|$ is small. Thus $\{z_k(t)\}$ converges in C^1 on $|t| \leqslant |t^*|$. This proves Lemma 3.3.

We interpret this lemma in terms of chains. We seek a chain from the origin to some nearby point (z^*, u^*). We identify u with t. Thus we seek to choose some ν for which the solution to (3.3), (3.4) satisfies $z(t^*) = z^*$. Now we have just seen that the boundary problem (3.11) can be solved under certain restrictions on (z^*, t^*). The solution to this boundary problem determines ν by (3.6) and, with the appropriate μ, solves (3.3). This provides the desired chain. Thus the lemmas in this section establish the following result.

Proposition. *There is some positive function $\delta(R)$ such that each point in the set*

$$T(R) = \{(z^*, u^*): |u^*| \geqslant |z^*|/R, |z^*| + |u^*| \leqslant \delta(R)\}$$

can be connected to the origin by some chain.

4. We now outline the proof of our theorem for higher dimensional CR structures. Since not every CR structure is realizable (see [5] and [6]) we shall also drop the assumption that we are dealing with a submanifold of a complex manifold. So we start with the definition of an abstract CR manifold. (To be precise, we define an integrable CR manifold of hypersurface type.)

Definition. Let V be a subbundle of the complexified tangent bundle of M^{2n+1}. Then (M, V) is called a CR structure if

(1) $\dim_{\mathbf{C}} V = n$,

(2) $V \cap \overline{V} = \{0\}$,

(3) $[V, V] \subset V$.

This last condition means that the Lie bracket of any two sections of V is itself a section of V. (This does not imply that V arises from a submanifold of M. There is a great difference between the real and complex Frobenius theorems.) With V understood we say that M is a CR manifold.

Let $\theta, \theta^1, \cdots, \theta^n$ be a set of one-forms satisfying

(a) θ is real and $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0$,

(b) $\theta(L) = \theta(\overline{L}) = 0$ and $\theta^\alpha(L) = 0$ for every $L \in V$ and $\alpha = 1, \cdots, n$.

Then is it clear that $d\theta = 0$ and $d\theta^\alpha = 0$, both $\operatorname{mod}\{\theta, \theta^1, \cdots, \theta^n\}$. In particular, since θ is real

$$d\theta = i g_{\alpha\beta} \theta^\alpha \cap \overline{\theta^\beta} \quad \operatorname{mod}\{\theta\},$$

where g is a hermitian matrix.

The choice of such one-forms is not unique. But the signature of g does not depend on the choice and so the following definition is meaningful.

Definition. (M, V) is said to be strictly pseudo-convex if g is positive definite.

In this case we may choose our forms θ and θ^α such that

$$d\theta = i\theta^\alpha \wedge \overline{\theta^\alpha} + \theta \wedge \tilde{\phi}. \tag{4.1}$$

Cartan's construction for three-dimensional CR structures has been extended to higher dimensions by Chern and by Tanaka. See the survey article [1] for these and other references. In particular there are unique forms ϕ^α_β and $\tilde{\phi}^\alpha$ on M such that $\{\theta, \theta^\alpha, \tilde{\phi}, \phi^\alpha_\beta, \tilde{\phi}^\alpha\}$ satisfy certain equations. A chain (together with a choice of parametrization) is a curve $\gamma(t) \subset M$ satisfying

$$\begin{aligned} &\theta(\dot{\gamma}(t)) = 1, \qquad \theta^\alpha(\dot{\gamma}(t)) = -\mu^\alpha(t), \\ &\frac{d\mu^\alpha}{dt} = i\mu^\alpha|\mu|^2 + \frac{1}{2}\tilde{\phi}(\dot{\gamma}(t))\mu^\alpha + \mu^\beta\phi^\alpha_\beta(\dot{\gamma}(t)) + \tilde{\phi}^\alpha(\dot{\gamma}(t)). \end{aligned} \tag{4.2}$$

See [1] for details; we have changed their equations on p. 105 by setting $\mu^\alpha = -2a^\alpha$, reversing the signs of θ and t, and using $+i$ in (4.1). This was done to obtain the same signs as in (0.10). It is a consequence of the Cartan-Chern-Tanaka construction that these curves do not depend on the original choice of θ, θ^α, $\tilde{\phi}$.

The hyperquadric Q in $\mathbf{C}^n$ is given by

$$\operatorname{Im} z_{n+1} = \sum_1^n |z_\alpha|^2.$$

Local coordinates on this surface are given by $(z_1, \cdots, z_n, u)$ with $u = \operatorname{Re} z_{n+1}$. We may take

$$\theta = \tfrac{1}{2}(du + iz^\alpha d\bar{z}^\alpha - i\bar{z}^\alpha dz^\alpha), \quad \theta^\alpha = dz^\alpha, \qquad \alpha = 1, \cdots, n.$$

Then the equations for chains are

$$\theta(\dot{\gamma}) = 1, \quad \theta^\alpha(\dot{\gamma}) = -\mu^\alpha, \quad \frac{d\mu^\alpha}{dt} = i\mu^\alpha|\mu|^2.$$

It follows as in §1 that the chains are the intersections of Q with complex lines. In particular any two points may be joined by a chain. A local version holds for any strictly pseudo-convex CR structure; in this section we will sketch the proof.

Theorem. *On a strictly pseudo-convex CR manifold any two sufficiently close points can be connected by a chain.*

In fact chains can be defined on any CR manifold provided g is nonsingular. However, it is no longer true that nearby points must lie on a common chain. Consider the hyperquadric

$$Q_{1,1} = \left\{(z_1, z_2, z_3) \colon \operatorname{Im} z_3 = |z_1|^2 - |z_2|^2\right\}.$$

The chains are again the intersections of $Q_{1,1}$ with complex lines. Thus a chain connecting $(0,0,0)$ and $(1,1,0)$ would have to be given by $Q_{1,1} \cap \{(z, z, 0)\colon z \in \mathbf{C}\}$. But $\{(z, z, 0)\} \subset Q_{1,1}$ and the intersection is thus not a real curve. Of course it is still possible that the theorem holds for a generic nonsingular CR manifold. It is easy to see why our proof breaks down without strict pseudo-convexity. For example, for the chains on $Q_{1,1}$ one has $d\mu_j = i\mu_j\|\mu\|^2\alpha$, with $\|\mu\|^2 = |\mu_1|^2 - |\mu_2|^2$ and so $\|\mu\|^{-2}$ cannot be used to define a new time parameter.

To start the proof of the Theorem we will write (4.2) in a form similar to (2.2) and obtain estimates like (2.3)–(2.7). To do this we again introduce special local coordinates and choose particular one-forms. Let p be a given point of M^{2n+1}.

Lemma 4.1. *We can choose local coordinates $z^1, \cdots, z^n$, u and forms $\theta, \theta^1, \cdots, \theta^n$ such that*

(a) *p becomes the origin,*

(b) *θ and each θ^α annihilate V,*

(c) *θ is real and $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0$,*

(d) *$d\theta = i\theta^\alpha \wedge \overline{\theta^\alpha} \pmod{\theta}$,*

(e) *$\theta = \frac{1}{2}(du + (iz^\alpha - b^\alpha)\, d\bar{z}^\alpha + (-i\bar{z}^\alpha - \bar{b}^\alpha)\, dz^\alpha)$ with $b^\alpha = \mathcal{O}(3)$,*

(f) *$\theta^\alpha = dz^\alpha + b^\alpha_\beta\, dz^\beta + a^\alpha_\beta\, d\bar{z}^\beta$ with $b^\alpha_\beta = \mathcal{O}(2)$ and $a^\alpha_\beta = \mathcal{O}(2)$.*

In (f) we can actually achieve $a^\alpha_\beta = \mathcal{O}(N)$ for all N, but $N = 2$ suffices for our purposes.

The proof uses the formal solution to the CR embedding problem and a weak version of the normal form from [3].

We use $\theta, \theta^1, \cdots, \theta^n$ in (4.2) to write down the equations for chains. We do this in our local coordinates. We also want to change our time parameter. So write (4.2) as $\theta(\gamma_s(s)) = 1$, etc., and along each chain set $s = \int_0^t |\mu(\tau)|^{-2}\, d\tau$. Denote t derivatives by $\dot{\gamma}$, etc. Then from (4.2) and the above lemma we obtain

$$\dot{u} = 2/|\mu|^2 - iz^\alpha \dot{\bar{z}}^\alpha + i\bar{z}^\alpha \dot{z}^\alpha + b^\alpha \dot{\bar{z}}^\alpha + \bar{b}^\alpha \dot{z}^\alpha, \tag{4.3}$$

$$\dot{z}^\alpha + b^\alpha_\beta \dot{z}^\beta + a^\alpha_\beta \dot{\bar{z}}^\beta = -\mu^\alpha/|\mu|^2, \tag{4.4}$$

$$\dot{\mu}^\alpha = i\mu^\alpha + \tfrac{1}{2}\mu^\alpha \tilde{\phi}(\dot{z}, \dot{u}) + \mu^\beta \phi^\alpha_\beta(\dot{z}, \dot{u}) + \tilde{\phi}^\alpha(\dot{z}, \dot{u}). \tag{4.5}$$

Using (4.4) and its conjugate we see that

$$\dot{z}^\alpha = -\mu^\alpha/|\mu|^2 + C^\alpha_\beta \mu^\beta/|\mu|^2 + e^\alpha_\beta \bar{\mu}^\beta/|\mu|^2 \tag{4.6}$$

with $C^\alpha_\beta = \mathcal{O}(2)$ and $e^\alpha_\beta = \mathcal{O}(2)$. Thus we may write

$$\dot{z}^\alpha = -\mu^\alpha/|\mu|^2 + P^\alpha(z, u, \mu), \tag{4.7}$$

where, as long as $|z| + |u| < C_1$ and $|\mu| > C_2$,

$$|P| < C\big(|z|^2 + |u|^2\big)/|\mu|,$$
$$\|P_z\| + |P_u| < C(|z| + |u|)/|\mu|,$$
$$\|P_\mu\| < C\big(|z|^2 + |u|^2\big)/|\mu|^2.$$

We use the same norm notation as previously but now for vectors.

Using (4.7) in (4.3) we see that

$$\dot{u} = 2/|\mu|^2 + \mathcal{O}(1/|\mu|). \tag{4.8}$$

It follows that (4.5) can be rewritten as

$$\dot{\mu}^\alpha = i\mu^\alpha + B^\alpha(z, u, \mu). \tag{4.9}$$

As long as $|z| + |u| < C_1$ and $|\mu| > C_2$ we have $|B| + \|B_z\| + |B_u| < C$ and $\|B_\mu\| < C/|\mu|$.

We want to compare a solution to (4.7), (4.8), (4.9) to the corresponding solution for the hyperquadric. The initial value problem on the hyperquadric is given by

$$\begin{aligned} &\dot{u} = 2/|\mu|^2 - iz^\alpha \dot{\bar{z}}^\alpha + i\bar{z}^\alpha \dot{z}^\alpha, \quad \dot{z}^\alpha = -\mu^\alpha/|\mu|^2, \quad \dot{\mu}^\alpha = i\mu^\alpha, \\ &z(0) = u(0) = 0, \qquad \mu(0) = \nu \end{aligned} \tag{4.10}$$

and on our general CR manifold by

$$\begin{aligned} &\dot{\eta} = 2/|\sigma|^2 - i\zeta^\alpha \dot{\bar{\zeta}}^\alpha + i\bar{\zeta}^\alpha \dot{\zeta}^\alpha + \beta^\alpha(\zeta, \eta)\dot{\zeta} + \bar{\beta}^\alpha \dot{\bar{\zeta}}^\alpha, \\ &\dot{\zeta}^\alpha = -\sigma^\alpha/|\sigma|^2 + P^\alpha(z, u, \mu), \qquad \dot{\sigma}^\alpha = i\sigma^\alpha + B^\alpha(z, u, \mu), \\ &\zeta(0) = \eta(0) = 0, \qquad \sigma(0) = \nu. \end{aligned} \tag{4.11}$$

In (4.11) we have the following estimates valid on some set $|\zeta| + |\eta| < \varepsilon^*$ and $|\sigma| > C^*$:

$$\begin{aligned}
&|\beta| < C\big(|\zeta|^3 + |\eta|^3\big),\\
&\|\beta_\zeta\| + |\beta_\eta| < C\big(|\zeta|^2 + |\eta|^2\big),\\
&|P| < C\big(|z|^2 + |u|^2\big)/|\sigma|,\\
&\|P_\zeta\| + |P_\eta| \leqslant C(|z| + |u|)/|\sigma|,\\
&\|P_\sigma\| \leqslant C\big(|z|^2 + |u|^2\big)/|\sigma|^2,\\
&|B| + \|B_\zeta\| + |B_\eta| < C,\\
&\|B_\sigma\| < C/|\sigma|.
\end{aligned}$$

We now indicate how the estimates in §2 can also be established for these equations. Again let $X(t) = (z(t), u(t))$ be a chain on the hyperquadric and $Y(t) = (\zeta(t), \eta(t))$ the corresponding chain on M^{2n+1}. It is easily seen that Lemmas 2.2, 2.3 and 2.4 remain true. In place of (2.11) and (2.12) we derive

$$\left|\frac{\sigma^\alpha}{|\sigma|^2} - \frac{\mu^\alpha}{|\mu|^2}\right| \leqslant \frac{C|t|}{|\nu|^2},$$

$$\left\|\left(\frac{\sigma^\alpha}{|\sigma|^2} - \frac{\mu^\alpha}{|\mu|^2}\right)_\nu\right\| \leqslant \frac{C|t|}{|\nu|^3},$$

while (2.13) and (2.14) remain valid. The rest of the proof of Lemma 2.1 then proceeds as before.

Next we need to compute the Jacobian of the transformation $(t, \nu) \to (\zeta(\nu, t), \eta(\nu, t))$. Let us use the following notation. Consider the column vectors in $\mathbf{C}^n \times R$ whose transposes are given by

$$\begin{aligned}
(E^k)^T &= \big(\zeta_1^k, \zeta_{\bar 1}^k, \cdots, \zeta_n^k, \zeta_{\bar n}^k, \zeta_t^k\big),\\
(E^{\bar k})^T &= \big(\overline{\zeta_1^k}, \cdots, \overline{\zeta_{\bar n}^k}, \overline{\zeta_t^k}\big),\\
F^T &= \big(\eta_1, \eta_{\bar 1}, \cdots, \eta_n, \eta_{\bar n}, \eta_t\big).
\end{aligned}$$

Then $D = \det(E^1, E^{\bar 1}, \cdots, E^n, E^{\bar n}, F)$ is the Jacobian determinant we seek to evaluate. Let D_Q be the determinant for the hyperquadric. That is, D_Q results when in place of ζ^α and η we use z^α and u. As is easily verified

$$z^\alpha(\nu, t) = \frac{i\nu^\alpha}{|\nu|^2}(e^{it} - 1), \qquad u(\nu, t) = \frac{2}{|\nu|^2}\sin t. \tag{4.12}$$

Thus $z^\alpha(\nu, t) = \nu^\alpha h(t) u(\nu, t)$ with $h(t) = \frac{i}{2}(e^{it} - 1)(\sin t)^{-1}$. This observation will simplify our calculation.

Lemma 4.2. $D_Q = 2^{n+1}(1 - \cos t)^n / |\nu|^{4n+2}$.

Proof. For D_Q we have

$$E^k - \nu^k hF = V_1, \qquad E^{\bar{k}} - \bar{\nu}^k \bar{h} F = V_2,$$

where V_1 is a column vector each component of which is zero except for $v_k = \overline{h(t)u(\nu, t)}$ and $v_{2n+1} = \nu^k \overline{h_t(t)} u(\nu, t)$. Similarly, V_2 is zero except for $v_{\bar{k}} = \overline{h(t)}u(t, \nu)$ and $v_{2n+1} = \bar{\nu}^k \overline{h_t(t)} u(\nu, t)$. Thus

$$\begin{aligned} D &= \det(E^1 - \nu^1 hF,\ E^{\bar{1}} - \bar{\nu}^1 \bar{h} F, \cdots, E^{\bar{n}} - \bar{\nu}^n \bar{h} F,\ F) \\ &= \det\begin{pmatrix} M & N \\ P & q \end{pmatrix}, \end{aligned}$$

where M is the diagonal matrix with entries $(uh, u\bar{h}, \cdots, uh, u\bar{h})$, $P = (\nu^1 h_t u, \bar{\nu}^1 \bar{h}_t u, \cdots, \bar{\nu}^n \bar{h}_t u)$ and $(N^T, q) = F^T = (u_1, u_{\bar{1}}, \cdots, u_n, u_{\bar{n}}, u_t)$. It is easy to see that

$$\begin{aligned} \det\begin{pmatrix} M & N \\ p & q \end{pmatrix} &= m_1 \cdots m_{2n} q - \sum_{j=1}^{2n} m_1 \cdots \hat{m}_j \cdots m_{2n} p_j n_j \\ &= u^{2n} |h|^{2n} u_t - u^{2n} |h|^{2n-2} \sum_{j=1}^{n} \left\{ h \bar{\nu}^j \bar{h}_t u_j + \bar{h} \nu^j h_t u_j \right\} \\ &= u^{2n} |h|^{2n} \left(u_t - 2\,\mathrm{Re}\left(\frac{h_t}{h} \sum_{j=1}^{n} \nu^j u_j \right) \right). \end{aligned}$$

The proof of the lemma is completed by a simple computation which makes use of the homogeneity of u with respect to ν.

Now we turn to the Jacobian for our general CR structure.

Lemma 4.3. *For* $|t| \leqslant 3\pi/2$ *and* $|\nu| > C_1$ *we have*

$$|D - D_Q| < C|t|^{2n+1} / |\nu|^{4n+3}.$$

Proof. We know from our basic estimates that

$$(4.13) \qquad \begin{aligned} |\zeta_\nu^\alpha - z_\nu^\alpha| &< C|t|^2/|\nu|^3, \qquad & |\zeta_t^\alpha - z_t^\alpha| &< C|t|/|\nu|^2, \\ |\eta_\nu - u_\nu| &< C|t|^2/|\nu|^4, \qquad & |\eta_t - u_t| &< C|t|/|\nu|^3. \end{aligned}$$

Further, as is easily seen from (4.12),

$$(4.14) \qquad \begin{aligned} |z_\nu| &< C|t|/|\nu|^2, \qquad & |z_t| &< C/|\nu|, \\ |u_\nu| &< C|t|/|\nu|^3, \qquad & |u_t| &< C/|\nu|^2. \end{aligned}$$

So

$$D = \det\left\{ \begin{pmatrix} A & U \\ B & V \end{pmatrix} + \begin{pmatrix} a & u \\ b & v \end{pmatrix} \right\}$$

with

$$|A_{ij}| < C|t|/|\nu|^2, \quad |B_i| < C/|\nu|, \quad |U_i| < C|t|/|\nu|^3, \quad |V| < C/|\nu|^2,$$

$$|a_{ij}| < C|t|^2/|\nu|^3, \quad |b_i| < C|t|/|\nu|^2, \quad |u_i| < C|t|^2/|\nu|^4, \quad |v| < C|t|/|\nu|^3.$$

Thus

$$|D - D_Q| = \left|D - \det\begin{pmatrix} A & U \\ B & V \end{pmatrix}\right| < C|t|^{2n+1}/|\nu|^{4n+3}$$

and we are done.

We can now show that the Proposition at the end of §2 also holds in higher dimension. It is clear that the flow associated to the chains can be compactified by embedding in a flow on a RP^{2n} – bundle over M^{2n+1} in the same way that we compactified the flow on M^3. However, let us not do this but rather sketch the alternate approach alluded to in §1.

So let (z^*, u^*) be a given point with $|u^*| \leqslant |z^*|/R$ and $|z^*| \leqslant 1/R$. We claim that if R is large enough, then there is a chain starting at the origin and going through (z^*, u^*). The result is true for Q, so let $z(\nu^*, t^*) = z^*$ and $u(\nu^*, t^*) = u^*$ for z and u given by (4.12). We shall again assume $0 < t^* \leqslant \pi$. The modifications necessary for $t^* < 0$ will be obvious.

Now define $F = \{(z^*, u^*, \sigma)\colon \sigma \in \mathbf{C}\}$ and $N = \{(Y(\nu, t), \sigma(\nu, t))\colon (Y, \sigma)$ solves (4.11), $0 \leqslant t \leqslant t^* + \pi/4$, $R \leqslant |\nu| \leqslant (|z^*|^2 + |u^*|^2)^{-1}\}$ for the given (z^*, u^*). Note that $|t| \leqslant 3\pi/2$ so all our previous estimates apply to those orbits in N. Note also that once we exclude the trivial case of $(z^*, u^*) = (0, 0)$ we have that N is compact. The boundary of N is contained in

$$\begin{aligned}
&\{(Y(\nu, t), \sigma(\nu, t))\colon 0 \leqslant t \leqslant 3\pi/2, |\nu| = R\} \\
&\quad \cup \left\{(Y(\nu, t), \sigma(\nu, t))\colon 0 \leqslant t \leqslant 3\pi/2, |\nu| = \left(|z^*|^2 + |u^*|^2\right)^{-1}\right\} \\
&\quad \cup \{(0, \nu)\colon \nu \in \mathbf{C}\} \\
&\quad \cup \left\{\left(Y(\nu, t^* + \pi/4), \sigma(\nu, t^* + \pi/4)\right)\colon R \leqslant |\nu|\right\}.
\end{aligned}$$

The following results show that F does not intersect the boundary of N.

Lemma 4.4. *There exists some R_0 such that if for some (R, t, ν, z^*, u^*) one has $Y(\nu, t) = (z^*, u^*)$ with*

$$|u^*| \leqslant |z^*|/R, \qquad |z^*| \leqslant 1/R, \tag{4.15}$$

$$|t| \leqslant 3\pi/2, \quad |\nu| \geqslant R_0, \quad R \geqslant R_0, \tag{4.16}$$

then one also has

$$R < |\nu| < \left(|z^*|^2 + |u^*|^2\right)^{-1}. \tag{4.17}$$

Lemma 4.5. *There exists some R_0 such that if (z^*, u^*) satisfies* (4.15) *for some $R \geqslant R_0$, then*

$$\left|Y(\nu, t^* + \pi/4) - (z^*, u^*)\right| > 0$$

for all ν, $|\nu| \geqslant R$.

Proof of Lemma 4.4. For any solution $Y(\nu, t) = (\zeta, \eta)$ we have $|\zeta| < C|\nu|^{-1}$ for $|t| \leqslant 3\pi/2$. Thus when $\zeta(\nu, t) = z^*$ we have $|z^*| < C|\nu|^{-1}$. But if we had $|\nu| \geqslant (|z^*|^2 + |u^*|^2)^{-1}$ we would have $|\nu| \geqslant C|\nu|^2$ which contradicts $|\nu| \geqslant R_0$, R_0 large. This establishes the right inequality in (4.17).

Next note that

$$|z(\nu, t)| = |z(\nu, t) - \zeta(\nu, t) + z^*| \leqslant C/|\nu|^2 + |z|^*$$

and so

$$|z(\nu, t)| \leqslant 1/R + C/|\nu|^2. \tag{4.18}$$

Similarly

$$|u(\nu, t)| \leqslant |z(\nu, t)|/R + Ct^2/|\nu|^3 + Ct^2/R|\nu|^2. \tag{4.19}$$

Let us consider separately the cases $|t| \leqslant \pi/2$ and $\pi/2 \leqslant |t| \leqslant 3\pi/2$. Note that in the first case, for example

$$2|\sin t| - Ct^2/|\nu| - Ct^2/R > \sqrt{2}\,|\sin t| \tag{4.20}$$

provided R_0 is large compared to C. Substituting the explicit formulas (4.12) into (4.19) we obtain

$$\frac{2}{|\nu|^2}|\sin t| \leqslant \frac{1}{R}(2(1 - \cos t))^{1/2}\left(\frac{1}{|\nu|}\right) + \frac{Ct^2}{|\nu|^3} + \frac{Ct^2}{R|\nu|^2}$$

which implies, because of (4.20),

$$\frac{|\nu|}{R} > \frac{|\sin t|}{(1 - \cos t)^{1/2}} = (1 + \cos t)^{1/2} \geqslant 1.$$

So $|\nu| > R$ in the case when $|t| \leqslant \pi/2$.

In the other case, $\pi/2 \leqslant |t| \leqslant 3\pi/2$, we note that

$$\sqrt{2}\,(1 - \cos t)^{1/2} - C/|\nu| > 1 \tag{4.21}$$

provided $|\nu| > R_0$, R_0 large compared to C. We now substitute (4.12) into (4.18) to obtain

$$\frac{1}{|\nu|}(2(1 - \cos t))^{1/2} \leqslant \frac{1}{R} + \frac{C}{|\nu|^2}$$

and so (4.21) implies $|\nu| > R$ in the case where $\pi/2 \leqslant |t| \leqslant 3\pi/2$. Thus the left inequality of (4.17) is also valid.

Proof of Lemma 4.5. Set $z(\nu^*, t^*) = z^*$ and $u(\nu^*, t^*) = u^*$. Let $T = t^* + \pi/4$. So

$$|\zeta(\nu, T) - z^*| = |\zeta(\nu, T) - z(\nu, T)| + |z(\nu, T) - z^*|$$
$$\geqslant |z(\nu, T) - z^*| - C/|\nu|^2;$$

hence

$$|\zeta(\nu, T) - z^*| \geqslant |z(\nu, T) - z(\nu^*, t^*)| - C/|\nu|^2,$$

and similarly

$$|\eta(\nu, T) - u^*| \geqslant |u(\nu, T) - u(\nu^*, t^*)| - C/|\nu|^3.$$

Thus it suffices to show that either

$$\left|\frac{\nu}{|\nu|^2}(e^{iT} - 1) - \frac{\nu^*}{|\nu^*|^2}(e^{it^*} - 1)\right| - \frac{C}{|\nu|^2} > 0 \tag{4.22}$$

or

$$\left|\frac{\sin T}{|\nu|^2} - \frac{\sin t^*}{|\nu^*|^2}\right| - \frac{C}{|\nu|^3} > 0. \tag{4.23}$$

Let

$$E(\nu, \nu^*, t^*) = \left|\frac{\nu}{|\nu|}(e^{iT} - 1) - \frac{\nu^*}{|\nu^*|}\frac{|\nu|}{|\nu^*|}(e^{it^*} - 1)\right| + \left|\sin T - \frac{|\nu^2|}{|\nu^*|^2}\sin t^*\right|.$$

Note that if $E(\nu, \nu^*, t^*) > C^*$ for some $C^* > 0$ and all ν, ν^*, t^*, then either (4.22) or (4.23) must also hold provided only that $|\nu| \geqslant R_0$ where R_0 depends only on C^*.

For any unit vector $w \in \mathbf{C}^n$ we have $E \geqslant |a - rf| + |b - r^2 g|$ with

$$a = \frac{\nu \cdot \overline{w}}{|\nu|}(e^{iT} - 1), \quad b = \sin T, \quad f = \frac{\nu^* \cdot \overline{w}}{|\nu^*|}(e^{it^*} - 1),$$
$$g = \sin t^*, \quad r = \frac{|\nu|}{|\nu^*|}.$$

We want to apply Lemma 2.7 and we need to obtain that the lower bound C which it provides does not depend on ν or ν^*. So let us first choose the unit vector w such that

$$\left|\frac{\nu}{|\nu|} \cdot \overline{w}\right| = \left|\frac{\nu^*}{|\nu^*|} \cdot \overline{w}\right| \geqslant \frac{1}{\sqrt{2}}.$$

Then $|a|$ is bounded from below independently of ν and $|f(t^*)|$ is bounded from below on any interval $\delta \leqslant |t^*| \leqslant 5\pi/4$ independently of ν^*. Further

$$\left|a(t^*)/f(t^*)\right|^2 = (1 - \cos T)/(1 - \cos t^*)$$

and so

$$\left|b(t^*) - \left|a(t^*)/f(t^*)\right|^2 g(t^*)\right| \geqslant C_1 > 0.$$

Thus for each value of ν and ν^* we may apply Lemma 2.7 to obtain that $E(\nu, \nu^*, t^*) > C$, where C does not in fact depend on ν or ν^*. So we are done.

Now we again use the deformation argument to show that N and F do intersect. So let (ω, ω^α) be the standard CR structure on Q and (θ, θ^α) the given CR structure on M as in Lemma 4.1. We cannot use the simple linear deformation

$$\psi = (1-s)\omega + s\theta, \qquad \psi^\alpha = (1-s)\omega^\alpha + s\theta^\alpha$$

because (ψ, ψ^α) is in general not integrable and so the Cartan-Chern-Tanaka construction does not apply. Instead we consider for each $s > 0$ the map $\Phi(Z, U)$ given by $z = sZ$, $u = s^2 U$ and define the forms

$$\theta_s = \Phi^*(\theta)/s^2, \qquad \theta_s^\alpha = \Phi^*(\theta^\alpha)/s.$$

For $s = 0$ we set

$$\theta_0 = \tfrac{1}{2}(dU + iZ^\alpha d\bar{Z}^\alpha - i\bar{Z}^\alpha dZ^\alpha), \qquad \theta_0^\alpha = dZ^\alpha.$$

Lemma 4.6. *The forms $\theta_s, \theta_s^1, \cdots, \theta_s^n$, and all their derivatives with respect to Z and U, are continuous functions on $s \geqslant 0$. These forms define a strictly pseudo-convex (integrable) CR structure.*

We omit the simple proof.

Now let $N^s = \{Y^s(\nu, t), \sigma^s(\nu, t)\colon (Y^s, \sigma^s)$ solve (4.11) for the CR structure θ_s, θ_s^α, $0 \leqslant t \leqslant t^* + \pi/4$, $R \leqslant |\nu| \leqslant (|z^*|^2 + |u^*|^2)^{-1}\}$. Since we have uniform bounds on all coefficients and their derivatives, the largeness condition on R can be taken independent of s. Then we know that

(a) $N^0 \cap F$ is nonempty,

(b) $(\operatorname{bdy} N^s) \cap F$ is empty, $0 \leqslant s \leqslant 1$,

(c) if $p \in N^s \cap F$ for some s and some p, then N^s and F are transverse at p.

It follows that $N^12 \cap F$ is also nonempty. Thus there is some chain which connects the origin and (z^*, u^*). Hence the Proposition at the end of §2 also holds in higher dimensions and for abstract CR manifolds.

Now we turn to the results in §3 which fortunately can be easily generalized. (The estimates we will obtain are somewhat different from those used in §3 mainly because of the absence of b^{α}_{β} in the lowest dimensional case.) First we introduce $t \equiv u$ as a new time variable. Equations (4.2) then become in our special coordinates and after some simplification

$$\dot{z} = -\tfrac{1}{2}\mu + \mathscr{A}\dot{\bar{z}} + \mathscr{B}, \qquad \dot{\mu} = U(z,t,\mu). \tag{4.24}$$

We have

$$|\mathscr{A}| < C(|\mu|+1)(|t|+|z|), \qquad |\mathscr{B}| < C(|\mu|+1)^2(|t|+|z|),$$
$$|\mathscr{A}_\mu| < C(|t|+|z|), \qquad |\mathscr{B}_\mu| < C(|\mu|+1)(|t|+|z|)$$

and

$$U \text{ is smooth on } \{(z,t,\mu)\colon |t|+|z|<\varepsilon,\ \mu\in\mathbf{C}^n\}$$

provided $(|\mu|+1)(|t|+|z|) < 1$. Hence we have

$$\dot{z} = -\tfrac{1}{2}\mu + f(z,\bar{z},\mu,\bar{\mu},t) \tag{4.25}$$

with

$$|f| \leqslant C(|\mu|+1)^2(|t|+|z|), \qquad |f_\mu| \leqslant C(|\mu|+1)(|t|+|z|)$$

provided $(|\mu|+1)(|t|+|z|) < \varepsilon_1$ for some ε_1.

From this equation we derive

$$\mu = G(z,\dot{z},t) \tag{4.26}$$

provided ε_1 is small enough and $|\dot{z}|(|t|+|z|) < \varepsilon_2$ for some ε_2.

Then the second equation in (4.24) can be rewritten

$$\dot{\mu} = g(z,\dot{z},t). \tag{4.27}$$

This when substituted into the differentiated form of (4.25) yields $\ddot{z} = F(z,\dot{z},t)$. In this way we can obtain the following result.

Lemma 4.7. *There exist constants* ε *and* ε_2 *and smooth functions* $F(z,\bar{z},w,\bar{w},t)\in\mathbf{C}^n$ *and* $G(z,\bar{z},w,\bar{w},t)\in\mathbf{C}^n$ *on*

$$S = \{(z,w,t)\colon |t|+|z|<\varepsilon,\ |w|(|t|+|z|)<\varepsilon_2\}$$

such that if $z(t)\in\mathbf{C}^n$ *solves* $\ddot{z} = F(z,\bar{z},\dot{z},\dot{\bar{z}},t)$, $z(0)=0$, *then* $z(t)$ *together with* $\mu(t) = G(z(t),\bar{z}(t),\dot{z}(t),\dot{\bar{z}}(t),t)$ *solves* (4.24).

Recall we want to show that for each R there is some δ such that there is a chain from the origin to (z^*,u^*) whenever $|z^*| \leqslant R|u^*|$ and $|u^*|+|z^*|<\delta$. We now know this is equivalent to the following result.

Lemma 4.8. *Let* $F(z, \bar{z}, w, \bar{w}, t) \in \mathbf{C}^n$ *be smooth on*

$$S = \{(z, w, t) : |t| + |z| < \varepsilon, |w|(|t| + |z|) < \varepsilon_1\}$$

where ε *and* ε_1 *are given constants. For each* R *there is some* δ *such that*

$$\ddot{z} = F(z, \bar{z}, \dot{z}, \dot{\bar{z}}, t), \qquad z(0) = 0,\ z(t^*) = z^*,$$

has a solution provided

$$|z^*| < R|t^*|, \quad |z^*| + |t^*| < \delta.$$

The proof is exactly the same as the proof of Lemma 3.3 with vector notation now understood.

Thus the proposition at the end of §3 also is valid in higher dimensions and for abstract CR manifolds. This concludes our proof that any two sufficiently close points on a strictly pseudo-convex CR manifold lie on a common smooth chain.

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